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Oblique warped products

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Abstract

We define the oblique warped products and prove their existence. In addition to the Levi-Civita connection we use both the Schouten–Van Kampen and Vrănceanu connections to study the foliation and curvatures of an oblique warped product. As an application to cosmology we introduce the oblique Robertson–Walker spacetime and give its basic properties.

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0. Introduction

The notion of warped product has been introduced by Bishop and O'Neill [2] as a need for the construction of a large class of complete manifolds of negative curvature. Also, the warped products turned out to be the standard spacetime models of the universe as they are the simplest models of neighborhoods of stars and black holes (cf. O'Neill [4]).

The purpose of the present paper is to define and study a generalization of warped products. Let (E, h) and (F, k) be two semi-Riemannian manifolds and f be a positive smooth function on E. Then the warped product $M = E \times_f F$ is the product manifold $E \times F$ endowed with the semi-Riemannian metric

$$g = \pi^*h + (f \circ \pi)^2\sigma^*k,$$

where π and σ are the projections of $E \times F$ onto E and F respectively. One of the main properties of M is that the two factors (E, h) and (F, k) are orthogonal with respect to g. As a direct consequence of this fact, the warped product M is filled by two complementary orthogonal foliations: one is totally geodesic and the other is totally umbilical. We remove the condition for (E, h) and (F, k) to be orthogonal and obtain what we call a generalized warped product. More precisely, $M = E \times F_{(f,L)}$ is a generalized warped product if its semi-Riemannian metric g is given by (1.2). In particular, for $L = 0$ we obtain a warped product. For a non-zero L we call $M = E \times F_{(f,L)}$ an oblique warped product.

First, we study the existence of oblique warped products (see Theorem 1.2 and Corollary 1.2). Then we study the geometry of (M, g) by using the Schouten–Van Kampen and Vrănceanu connections induced by the Levi-Civita

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connection. The existence of the vertical foliation $\mathcal{F}_F$ and the (f, k)-associated distribution $\mathcal{D}$ enables us to develop an adapted tensor calculus which includes vertical and horizontal covariant derivatives. By using the above linear connections we characterize special classes of oblique warped products (cf. Theorems 2.2 and 2.3). Also, an interesting relation between the sectional curvatures of the distribution $\mathcal{D}$ is derived (cf. (3.24)), provided the semi-Riemannian metric of M is bundle-like for the vertical foliation. Finally, we introduce the oblique Robertson–Walker spacetime and apply the general theory that we developed in the previous sections to the case when an oblique warped product carries two complementary orthogonal foliations.

1. Oblique warped products

Let (E, h) and (F, k) be two semi-Riemannian manifolds. Consider the product manifold $M = E \times F$ and denote by π and σ the projections of M onto E and F respectively.

Throughout the paper all manifolds are paracompact, and mappings are smooth (differentiable of class C^∞). We denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth vector fields on M. We use similar notation for any other manifold or vector bundle. Also, we use the Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range. If not stated otherwise, throughout the paper we use the following ranges for indices: $i, j, k, \ldots \in \{1, \ldots, n\}$; $\alpha, \beta, \gamma, \ldots \in \{1, \ldots, p\}$.

Now, we denote by $\mathcal{D}_E$ and $\mathcal{D}_F$ the distributions on M that are tangent to the foliations whose leaves are $\{\pi^{-1}(p)\}_{p\in E}$ and $\{\sigma^{-1}(q)\}_{q\in F}$ respectively. As they are complementary distributions in TM we put

$$TM = \mathcal{D}_E \oplus \mathcal{D}_F. \tag{1.1}$$

In what follows we denote by the same symbols h and k the semi-Riemannian metrics on $\mathcal{D}_E$ and $\mathcal{D}_F$ defined by the semi-Riemannian metrics h and k on E and F respectively. Thus we have two complementary semi-Riemannian distributions $(\mathcal{D}_E, h)$ and $(\mathcal{D}_F, k)$ on M. So far, two semi-Riemannian structures have been considered on M. They are the semi-Riemannian product manifold and the warped product (cf. O'Neill [4], pp. 57, 205). A generalization of these structures is given in the present paper as follows.

Definition 1.1. Let f be a positive smooth function on E and $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$ be an $\mathcal{F}(M)$-bilinear mapping. Taking into account (1.1) we denote by P_E and P_F the projection morphisms of $\Gamma(TM)$ onto $\Gamma(\mathcal{D}_E)$ and $\Gamma(\mathcal{D}_F)$ respectively. Then we define the symmetric bilinear mapping $g : \Gamma(TM) \times \Gamma(TM) \longrightarrow \mathcal{F}(M)$ by

$$\begin{aligned} g(X, Y) = {} & h(P_E X, P_E Y) + (f \circ \pi)^2 k(P_F X, P_F Y) \\ & + L(P_E X, P_F Y) + L(P_E Y, P_F X), \quad \forall X, Y \in \Gamma(TM). \end{aligned} \tag{1.2}$$

If g is a semi-Riemannian metric on the product manifold $M = E \times F$ then we put $M = E \times F_{(f,L)}$ and call it a *generalized warped product*. For $L = 0$, M becomes a warped product with base E and fibre F. For a non-zero L we call $M = E \times F_{(f,L)}$ an *oblique warped product* with base E and fibre F. In this case, we also say that (h, k, f, L) is an *oblique warped product structure* on $E \times F$.

Next, in order to construct a large class of oblique warped products, we study the existence of a non-zero $\mathcal{F}(M)$-bilinear mapping L which has been used to define g by (1.2). Let p and n be the dimensions of E and F respectively. In what follows we take (x^α, y^i) as a coordinate system on $M = E \times F$, where (x^α), $\alpha \in \{1, \ldots, p\}$, and (y^i), $i \in \{1, \ldots, n\}$, are coordinate systems on E and F respectively. Then the coordinate transformations on M are given by

$$\text{(a) } \widetilde{x}^\alpha = \widetilde{x}^\alpha(x^1, \ldots, x^p), \qquad \text{(b) } \widetilde{y}^i = \widetilde{y}^i(y^1, \ldots, y^n). \tag{1.3}$$

Accordingly, the natural frame fields $\left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^i}\right\}$ and $\left\{\frac{\partial}{\partial \widetilde{x}^\beta}, \frac{\partial}{\partial \widetilde{y}^j}\right\}$ are related by

$$\text{(a) } \frac{\partial}{\partial x^\alpha} = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \frac{\partial}{\partial \widetilde{x}^\beta}, \qquad \text{(b) } \frac{\partial}{\partial y^i} = \frac{\partial \widetilde{y}^j}{\partial y^i} \frac{\partial}{\partial \widetilde{y}^j}. \tag{1.4}$$

Now, we can prove the following.

Lemma 1.1. *There exists a complementary distribution $\mathcal{D}$ to $\mathcal{D}_F$ in TM, if and only if on the domain of each local chart on M there exist np smooth functions D^i_α, $\alpha \in \{1, \ldots, p\}$, $i \in \{1, \ldots, n\}$, satisfying*

$$D^i_\alpha \frac{\partial \widetilde{y}^j}{\partial y^i} = \widetilde{D}^j_\beta \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha}, \tag{1.5}$$

with respect to the coordinate transformations (1.3).

Proof. First, suppose $\mathcal{D}$ is a complementary distribution to $\mathcal{D}_F$ in TM and take a local frame field $\left\{E_\alpha, \frac{\partial}{\partial y^i}\right\}$ on M such that $E_\alpha \in \Gamma(\mathcal{D})$, $\alpha \in \{1, \ldots, p\}$, and $\frac{\partial}{\partial y^i} \in \Gamma(\mathcal{D}_F)$, $i \in \{1, \ldots, n\}$. Then we put

$$\frac{\partial}{\partial x^\alpha} = C^\beta_\alpha E_\beta + D^i_\alpha \frac{\partial}{\partial y^i}, \tag{1.6}$$

where C^β_α and D^i_α are smooth functions on a coordinate neighbourhood in M. Thus the matrix of transition from $\left\{E_\alpha, \frac{\partial}{\partial y^i}\right\}$ to $\left\{\frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial y^j}\right\}$ is

$$\Lambda = \begin{bmatrix} C^\beta_\alpha & 0 \\ D^i_\alpha & \delta^i_j \end{bmatrix}.$$

Hence the $p \times p$ matrix $[C^\beta_\alpha]$ is non-singular since Λ is so. As a consequence it follows that $\mathcal{D}$ is also locally spanned by

$$\frac{\delta}{\delta x^\alpha} = C^\beta_\alpha E_\beta, \quad \alpha \in \{1, \ldots, p\}.$$

Thus (1.6) becomes

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - D^i_\alpha \frac{\partial}{\partial y^i}, \quad \alpha \in \{1, \ldots, p\}. \tag{1.7}$$

Moreover, by using (1.4) and (1.7) for two coordinate systems (x^α, y^i) and $(\widetilde{x}^\beta, \widetilde{y}^j)$ with overlapping domains, we obtain

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \frac{\partial}{\partial \widetilde{x}^\beta} - D^i_\alpha \frac{\partial \widetilde{y}^j}{\partial y^i} \frac{\partial}{\partial \widetilde{y}^j} = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \frac{\delta}{\delta \widetilde{x}^\beta} + \left(\frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \widetilde{D}^j_\beta - D^i_\alpha \frac{\partial \widetilde{y}^j}{\partial y^i} \right) \frac{\partial}{\partial \widetilde{y}^j}. \tag{1.8}$$

Hence we obtain (1.5) for the functions D^i_α from (1.6) and the $\frac{\delta}{\delta x^\alpha}$ given by (1.7) satisfy

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \frac{\delta}{\delta \widetilde{x}^\beta}, \qquad \alpha \in \{1, \ldots, p\}. \tag{1.9}$$

Conversely, suppose on the domain of each local chart on M there exist smooth functions D^i_α, $\alpha \in \{1, \ldots, p\}$, $i \in \{1, \ldots, n\}$, satisfying (1.5). Then by (1.7) we define $\left\{\frac{\delta}{\delta x^\alpha}\right\}$, $\alpha \in \{1, \ldots, p\}$, and by using (1.5) in (1.8) we obtain (1.9). This means that there exists on M a distribution $\mathcal{D}$ of rank p which is locally represented by $\frac{\delta}{\delta x^\alpha}$, $\alpha \in \{1, \ldots, p\}$, defined by (1.7). Thus the proof is complete. □

As (1.5) has a tensorial character we can state the following.

Corollary 1.1. *There exists a complementary distribution $\mathcal{D}$ to $\mathcal{D}_F$ in TM, if and only if there exists an $\mathcal{F}(M)$-linear mapping $L^* : \Gamma(\mathcal{D}_E) \longrightarrow \Gamma(\mathcal{D}_F)$.*

Next, suppose that M admits an $\mathcal{F}(M)$-bilinear mapping $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$. Then we put

$$L_{\alpha i} = L\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^i}\right), \tag{1.10}$$

and by using (1.4) we deduce that

$$L_{\alpha i} = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \frac{\partial \widetilde{y}^j}{\partial y^i} \widetilde{L}_{\beta j}, \tag{1.11}$$

with respect to the coordinate transformations (1.3). Conversely, suppose that on the domain of each local chart on M there exist np smooth functions $L_{\alpha i}$ satisfying (1.11) with respect to (1.3). Then we take $X \in \Gamma(\mathcal{D}_E)$ and $Y \in \Gamma(\mathcal{D}_F)$ and put

$$\text{(a) } X = X^\alpha \frac{\partial}{\partial x^\alpha}, \qquad \text{(b) } Y = Y^i \frac{\partial}{\partial y^i}. \tag{1.12}$$

By using (1.4) and (1.12) we infer that

$$X^\alpha \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} = \widetilde{X}^\beta, \qquad \text{(b) } Y^i \frac{\partial \widetilde{y}^j}{\partial y^i} = \widetilde{Y}^j. \tag{1.13}$$

Now, we define $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$ by

$$L(X, Y) = L_{\alpha i} X^\alpha Y^i. \tag{1.14}$$

By using (1.11) and (1.13) we deduce that (1.14) is invariant with respect to the coordinate transformation (1.3). Thus the above L is well defined. This discussion enables us to state the following.

Lemma 1.2. *There exists an $\mathcal{F}(M)$-bilinear mapping $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$, if and only if on the domain of each local chart on M there exist np smooth functions $L_{\alpha i}$, $\alpha \in \{1, \dots, p\}$, $i \in \{1, \dots, n\}$, satisfying* (1.11).

Now, let k_{ij} be the local components of the semi-Riemannian metric k on $\mathcal{D}_F$, that is, we have

$$k_{ij} = k\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right), \quad i, j \in \{1, \dots, n\}. \tag{1.15}$$

Then denote by k^{ij} the entries of the inverse matrix of $[k_{ij}]$ and by using (1.15) and (1.4b) we infer that

$$\text{(a) } k_{ij} = \frac{\partial \widetilde{y}^h}{\partial y^i} \frac{\partial \widetilde{y}^\ell}{\partial y^j} \widetilde{k}_{h\ell}, \qquad \text{(b) } k^{ij} \frac{\partial \widetilde{y}^h}{\partial y^i} \frac{\partial \widetilde{y}^\ell}{\partial y^j} = \widetilde{k}^{h\ell}. \tag{1.16}$$

Next, suppose $\mathcal{D} = (D^i_\alpha)$ is a complementary distribution to $\mathcal{D}_F$ in TM. Then locally, we define the functions

$$L_{\alpha i} = D^j_\alpha k_{ij} (f \circ \pi)^2, \tag{1.17}$$

and by using (1.5) and (1.16a) we obtain (1.11). Thus by Lemma 1.2 we obtain an $\mathcal{F}(M)$-bilinear mapping $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$. We call L the (f, k)-*associated mapping* to the distribution $\mathcal{D}$. Conversely, suppose L is given, and by using $L_{\alpha i}$ we define

$$D^i_\alpha = L_{\alpha j} k^{ij} \frac{1}{(f \circ \pi)^2}. \tag{1.18}$$

Then, by using (1.11) and (1.16b), we deduce that the D^i_α satisfy (1.5). Thus we obtain a complementary distribution $\mathcal{D}$ to $\mathcal{D}_F$ in TM, which we call the (f, k)-*associated distribution* to L. Thus we can state the following.

Theorem 1.1. *The semi-Riemannian metric $(f \circ \pi)^2 k$ on $\mathcal{D}_F$ defines, by* (1.17) *and* (1.18)*, a one-to-one mapping between the set of complementary distributions to $\mathcal{D}_F$ in TM and the set of $\mathcal{F}(M)$-bilinear mappings $\{L\}$.*

Remark 1.1. The (f, k)-associated distribution $\mathcal{D}$ to L is orthogonal to $\mathcal{D}_F$ with respect to g. Indeed, by using (1.7), (1.2), (1.10), (1.15) and (1.17) we obtain

$$g\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right) = L\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^i}\right) - D^j_\alpha k\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right)(f \circ \pi)^2 = L_{\alpha i} - D^j_\alpha k_{ji} (f \circ \pi)^2 = 0.$$

Proposition 1.1. *Let (E, h) and (F, k) be two semi-Riemannian manifolds, $M = E \times F$, and f a positive smooth function on E. Then there exists a non-zero $\mathcal{F}(M)$-bilinear mapping $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$.*

Proof. Let $z^a = (x^\alpha, y^i)$, $a \in \{1, \dots, n+p\}$ be the local coordinates on $\mathcal{U} \subset M$. Then define on $\mathcal{U}$ the frame field $\left\{X_\alpha, \frac{\partial}{\partial y^i}\right\}$ where $X_1 = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial y^1}$, $X_\alpha = \frac{\partial}{\partial x^\alpha}$, for $\alpha \neq 1$. Consider the standard Riemannian metric G on $\mathcal{U}$ given by

$$G(X_\alpha, X_\beta) = \delta_{\alpha\beta}, \qquad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \delta_{ij}, \qquad G\left(X_\alpha, \frac{\partial}{\partial y^i}\right) = 0,$$

and by using the partition of unity on M we extend it to the whole M. Then take the complementary orthogonal distribution $\mathcal{D}$ to $\mathcal{D}_F$ in TM with respect to G. Clearly, $\mathcal{D} \neq \mathcal{D}_E$ because $\frac{\partial}{\partial x^1} \notin \Gamma(\mathcal{D})$. Then the (f, k)-associated mapping L to $\mathcal{D}$ is the mapping we are looking for. □

Now, we are able to prove the existence of g from (1.2).

Theorem 1.2. *Let (E, h) be a 1-dimensional manifold with a negative definite metric h, (F, k) a Riemannian manifold and $M = E \times F$. Suppose that there exists a positive smooth function f on E. Then there exists a non-zero $\mathcal{F}(M)$-bilinear mapping $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$ such that M is an oblique warped product $(E \times F)_{(f,L)}$ with a Lorentz metric g given by (1.2).*

Proof. By Proposition 1.1 there exists a non-zero L such that g given by (1.2) is a symmetric $\mathcal{F}(M)$-bilinear mapping. So we only have to prove that g is non-degenerate metric of Lorentz signature. To this end we consider the (f, k)-associated distribution $\mathcal{D}$ to L. Thus $\mathcal{D}$ is a line distribution locally spanned by (see (1.7))

$$\frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1} - D_1^i \frac{\partial}{\partial y^i}, \tag{1.19}$$

where (x^1, y^i) are local coordinates on $M = E \times F$. Moreover, by Remark 1.1, $\mathcal{D}$ is complementary and orthogonal to $\mathcal{D}_F$ with respect to g. Now, we examine the restrictions of g to $\mathcal{D}$ and $\mathcal{D}_F$. First, g is positive definite on $\mathcal{D}_F$ because by (1.2) and (1.15) we have

$$g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = k_{ij}(f \circ \pi)^2,$$

and k is supposed to be positive definite. Then by using (1.19), (1.2), (1.10) and (1.17), and taking into account that $\mathcal{D}$ and $\mathcal{D}_F$ are orthogonal, we obtain

$$\begin{aligned} g\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^1}\right) &= g\left(\frac{\delta}{\delta x^1}, \frac{\partial}{\partial x^1}\right) \\ &= h\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) - D_1^i L\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^i}\right) \\ &= h\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) - k_{ij} D_1^i D_1^j (f \circ \pi)^2. \end{aligned}$$

Since h is negative definite and k is positive definite, we deduce that g is negative definite on $\mathcal{D}$. Therefore g is a Lorentz metric on M. □

Corollary 1.2. *Let I be an open interval in $\mathbb{R}$ endowed with a negative definite metric h, and (F, k) a Riemannian manifold. Then $M = I \times F$ can be equipped with an oblique warped product structure.*

Proof. Consider a positive function f on I and the assertion follows from Theorem 1.2. □

Remark 1.2. From the proof of Theorem 1.2 we see that given f, h, k and a non-zero L we can obtain an oblique warped product structure on $M = I \times F$. This enables us to construct concrete examples of oblique warped products and, in particular, to introduce in the last section what we call the oblique Robertson–Walker spacetime. □

Example 1.1. Let $E = I$ and $F = \mathbb{R}^n$ be equipped with the negative definite metric h and the Euclidean metric $k = (\delta_{ij})$, respectively. Take $(x^1, y^1, \ldots, y^n)$ as coordinates on $M = I \times \mathbb{R}^n$ and consider the distribution

$$\mathcal{D} = \text{span}\left\{\frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial y^1}\right\}.$$

Then it is easy to see that $\mathcal{D}$ is complementary to $\mathcal{D}_{\mathbb{R}^n}$ in TM, and according to (1.19) we have $D_1^i = \delta_1^i$. Next, we take a positive function f on I and by Theorem 1.1 obtain the non-zero (f, k)-associated mapping $L : \Gamma(\mathcal{D}_I) \times \Gamma(\mathcal{D}_{\mathbb{R}^n}) \longrightarrow \mathcal{F}(M)$ given by (see (1.17))

$$L_{1i} = \delta_{1i}(f \circ \pi)^2.$$

Thus M becomes an oblique warped product $I \times \mathbb{R}^n_{(f,L)}$ whose metric g with respect to the frame field $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^i}\right\}$ is given by the matrix

$$\begin{bmatrix} -1 & \delta_{1j}(f \circ \pi)^2 \\ \delta_{i1}(f \circ \pi)^2 & \delta_{ij}(f \circ \pi)^2 \end{bmatrix}. \quad \Box$$

Example 1.2. Let $\mathbb{R}^{2(q+1)}$ be the $2(q+1)$-dimensional Euclidean space endowed with the canonical complex structure J. Consider the $(2q+1)$-dimensional unit sphere S^{2q+1} embedded in $\mathbb{R}^{2(q+1)}$ and denote by N its unit normal vector field. Then $\xi = JN$ is a unit vector field that is tangent to S^{2q+1}. Take the local coordinates $(x^1, y^1, \ldots, y^{2q+1})$ on $M = I \times S^{2q+1}$ and consider the distribution

$$\mathcal{D} = \text{span}\left\{\frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1} - \xi^i\frac{\partial}{\partial y^i}\right\},$$

where (ξ^i), $i = \{1, \ldots, 2q+1\}$, are the local components of ξ. Clearly, $\mathcal{D}$ is complementary to $\mathcal{D}_{S^{2q+1}}$ and we have $D_1^i = \xi^i$. Now, take a negative definite metric h on I and consider S^{2q+1} endowed with the induced Riemannian metric $k = (k_{ij})$. Also, we put $\eta_i = k_{ij}\xi^j$ and obtain a globally defined 1-form $\eta = \eta_i dy^i$ on S^{2q+1}. Then for any positive function f on I we obtain the non-zero (f, k)-associated mapping L given by

$$L_{1i} = \eta_i(f \circ \pi)^2.$$

Thus M becomes an oblique warped product with the Lorentz metric g given locally by the matrix

$$\begin{bmatrix} -1 & \eta_j(f \circ \pi)^2 \\ \eta_i(f \circ \pi)^2 & k_{ij}(f \circ \pi)^2 \end{bmatrix}. \quad \Box$$

Example 1.3. Here we present a generalization of the previous examples. Let F be an n-dimensional Riemannian manifold which is either non-compact or it is compact and has Euler number $\chi(F) = 0$. Then by Proposition 37 on p. 149 in O'Neill [4] there exists on F a non-vanishing vector field ξ. From this point we follow the reasoning from Example 1.2 and obtain oblique warped product structures on $M = I \times F$. In particular, if F is a contact metric manifold then $M = I \times F$ admits oblique warped product structures. $\Box$

Any oblique warped product obtained from Example 1.3 will be called a *generic oblique warped product* (for short, *g.o.w. product*). To justify this name we show in the next example that g.o.w. products can be used to construct new examples of oblique warped products whose semi-Riemannian metrics are not necessarily of Lorentz type.

Example 1.4. Let $M = I \times F_{(f,L)}$ be a g.o.w. product with Lorentz metric $g = (g_{AB})$, $A, B \in \{1, \ldots, n+1\}$. Consider a $(p-1)$-dimensional manifold P endowed with a semi-Riemannian metric $\bar{h} = (\bar{h}_{ab})$, $a, b \in \{1, \ldots, p-1\}$. Take $Q = P \times I$ and define the function $\bar{f} : Q \longrightarrow R_+$ by $\bar{f}(p, t) = f(t)$, for any $(p, t) \in Q$. Also, we define $\overline{L} : \Gamma(\mathcal{D}_Q) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(Q \times F)$ by:

$$\overline{L}(X, Z) = 0 \quad \text{and} \quad \overline{L}(Y, Z) = L(Y, Z),$$

for any $X \in \Gamma(\mathcal{D}_P)$, $Y \in \Gamma(\mathcal{D}_I)$ and $Z \in \Gamma(\mathcal{D}_F)$. Then it is easy to see that $Q \times F_{(\bar{f},\overline{L})}$ is an oblique warped product whose semi-Riemannian metric has the matrix

$$\begin{bmatrix} \bar{h}_{ab} & 0 \\ 0 & g_{AB} \end{bmatrix}. \quad \square$$

The above construction of a semi-Riemannian oblique warped product enables us to state the following.

Proposition 1.2. *A semi-Riemannian product $P \times M$, where P is a semi-Riemannian manifold and M is a g.o.w. product, admits oblique warped product structures.*

2. The Schouten–Van Kampen and Vrănceanu connections

Let (E, h) and (F, k) be two semi-Riemannian manifolds of dimensions p and n respectively, f a positive smooth function on E, and $L : \Gamma(\mathcal{D}_E) \times \Gamma(\mathcal{D}_F) \longrightarrow \mathcal{F}(M)$ a non-zero $\mathcal{F}(M)$-bilinear mapping. Suppose that g from (1.2) is a semi-Riemannian metric on M and therefore $M = E \times F_{(f,L)}$ is an oblique warped product. Let $\mathcal{D}$ be the (f, k)-associated distribution to L, which is locally spanned by $\left\{\frac{\delta}{\delta x^\alpha}\right\}$, $\alpha \in \{1, \ldots, p\}$, given by (1.7).

We study the geometry of an oblique warped product by using the Levi-Civita connection and two other linear connections which have been used in the study of non-integrable distributions. The presentation of this study is using both methods: the local coordinates method and the coordinate-free method.

First, we consider the non-holonomic frame field $\left\{\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right\}$ on M, and by using (1.7), (1.2), (1.15) and (1.17) we obtain

$$\begin{aligned}
&\text{(a)} \quad g_{\alpha\beta}(x, y) = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) = h_{\alpha\beta}(x) - D^i_\alpha(x, y) D^j_\beta(x, y) k_{ij}(y) f^2(x),\\
&\text{(b)} \quad g'_{\alpha i}(x, y) = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right) = 0,\\
&\text{(c)} \quad \widetilde{g}_{ij}(x, y) = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = k_{ij}(y) f^2(x),
\end{aligned} \tag{2.1}$$

where $(x, y) = (x^\alpha, y^i)$, $(x) = (x^\alpha)$, $(y) = (y^i)$, $\alpha \in \{1, \ldots, p\}$, $i \in \{1, \ldots, n\}$, and

$$h_{\alpha\beta}(x) = h\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right). \tag{2.2}$$

To develop our study we use the concept of adapted tensor fields on M with respect to the decomposition (see Bejancu and Farran [1])

$$TM = \mathcal{D} \oplus \mathcal{D}_F. \tag{2.3}$$

First, we consider the Levi-Civita connection $\widetilde{\nabla}$ on (M, g) and according to (2.3) we put

$$\begin{aligned}
&\text{(a)} \quad \widetilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + G_\alpha{}^i{}_\beta \frac{\partial}{\partial y^i},\\
&\text{(b)} \quad \widetilde{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial y^i} = H_i{}^\gamma{}_\alpha \frac{\delta}{\delta x^\gamma} + K_i{}^j{}_\alpha \frac{\partial}{\partial y^j},\\
&\text{(c)} \quad \widetilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^\alpha} = L_\alpha{}^\gamma{}_i \frac{\delta}{\delta x^\gamma} + M_\alpha{}^j{}_i \frac{\partial}{\partial y^j},\\
&\text{(d)} \quad \widetilde{\nabla}_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = N_i{}^\gamma{}_j \frac{\delta}{\delta x^\gamma} + P_i{}^k{}_j \frac{\partial}{\partial y^k}.
\end{aligned} \tag{2.4}$$

Then by direct calculations we deduce that $(G_\alpha{}^i{}_\beta)$, $(H_i{}^\gamma{}_\alpha)$, $(L_\alpha{}^\gamma{}_i)$, $(M_\alpha{}^j{}_i)$ and $(N_i{}^\gamma{}_j)$ define adapted tensor fields, while $(F_\alpha{}^\gamma{}_\beta)$, $(K_i{}^j{}_\alpha)$ and $(P_i{}^k{}_j)$ are changed as follows:

$$F_\alpha{}^\gamma{}_\beta \frac{\partial \widetilde{x}^\nu}{\partial x^\gamma} = \widetilde{F}_\varepsilon{}^\nu{}_\mu \frac{\partial \widetilde{x}^\varepsilon}{\partial x^\alpha} \frac{\partial \widetilde{x}^\mu}{\partial x^\beta} + \frac{\partial^2 \widetilde{x}^\nu}{\partial x^\alpha \partial x^\beta},$$

$$K_i{}^j{}_\alpha \frac{\partial \widetilde{y}^h}{\partial y^j} = \widetilde{K}_k{}^h{}_\varepsilon \frac{\partial \widetilde{y}^k}{\partial y^i} \frac{\partial \widetilde{x}^\varepsilon}{\partial x^\alpha} - D^k_\alpha \frac{\partial^2 \widetilde{y}^h}{\partial y^k \partial y^i},$$
$$P_i{}^k{}_j \frac{\partial \widetilde{y}^h}{\partial y^k} = \widetilde{P}_\ell{}^h{}_m \frac{\partial \widetilde{y}^\ell}{\partial y^i} \frac{\partial \widetilde{y}^m}{\partial y^j} + \frac{\partial^2 \widetilde{y}^h}{\partial y^i \partial y^j}.$$

Now, we recall that $\widetilde{\nabla}$ is a torsion-free metric connection, that is, we have

$$\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

and

$$X(g(Y, Z)) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z), \quad \forall X, Y, Z \in \Gamma(TM). \tag{2.6}$$

Lemma 2.1. *Let $M = E \times F_{(f,L)}$ be an oblique warped product. Then we have the following:*

$$\begin{aligned}
&\text{(a)}\ \left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right] = I_\alpha{}^i{}_\beta \frac{\partial}{\partial y^i}, \quad \textit{where}\ \text{(b)}\ I_\alpha{}^i{}_\beta = \frac{\delta D^i_\alpha}{\delta x^\beta} - \frac{\delta D^i_\beta}{\delta x^\alpha},\\
&\text{(c)}\ \left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right] = D_i{}^j{}_\alpha \frac{\partial}{\partial y^j}, \quad \textit{where}\ \text{(d)}\ D_i{}^j{}_\alpha = \frac{\partial D^j_\alpha}{\partial y^i},\\
&\text{(e)}\ F_\alpha{}^\gamma{}_\beta = F_\beta{}^\gamma{}_\alpha, \qquad \text{(f)}\ G_\beta{}^i{}_\alpha - G_\alpha{}^i{}_\beta = I_\alpha{}^i{}_\beta, \qquad \text{(g)}\ H_i{}^\gamma{}_\alpha = L_\alpha{}^\gamma{}_i,\\
&\text{(h)}\ K_i{}^j{}_\alpha - D_i{}^j{}_\alpha = M_\alpha{}^j{}_i, \qquad \text{(i)}\ H_i{}^\gamma{}_\alpha = -g^{\gamma\beta} G_\beta{}^j{}_\alpha k_{ji} (f \circ \pi)^2,\\
&\text{(j)}\ N_i{}^\gamma{}_j = -g^{\gamma\alpha} M_\alpha{}^h{}_i k_{hj} (f \circ \pi)^2 = N_j{}^\gamma{}_i, \qquad \text{(k)}\ P_i{}^k{}_j = P_j{}^k{}_i,
\end{aligned} \tag{2.7}$$

where $g^{\gamma\alpha}$ are the entries of the inverse matrix of $[g_{\alpha\beta}]$.

Proof. By direct calculations using (1.7) we obtain (2.7a) and (2.7c). Next, both (2.7e) and (2.7f) are deduced from (2.5) by using (2.4a) and (2.7a). Similarly, by using (2.4b), (2.4c) and (2.7c) in (2.5) we derive (2.7g) and (2.7h). Now, by (2.6) and (2.1b) we have

$$g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^\beta}\right) + g\left(\frac{\partial}{\partial y^i}, \widetilde{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\delta}{\delta x^\beta}\right) = 0,$$

which implies (2.7i) via (2.4a), (2.4b) and (2.1). The first equality in (2.7j) follows in a similar way by using (2.4c) and (2.4d). Finally, (2.7k) and the second equality in (2.7j) are obtained by using (2.5) and (2.4d). □

Due to (2.7a) we can state the following.

Lemma 2.2. *The (f, k)-associated distribution $\mathcal{D}$ on the generalized warped product (M, g) is integrable if and only if $I_\alpha{}^i{}_\beta$ vanish identically on M for all $i \in \{1, \ldots, n\}$, $\alpha, \beta \in \{1, \ldots, p\}$.*

On the other hand, from (2.7f) we deduce that $I = (I_\alpha{}^i{}_\beta)$ is an adapted tensor field. Taking into account the above lemma we call I the *integrability tensor* of $\mathcal{D}$.

Next, we consider two linear connections on M with respect to which both distributions $\mathcal{D}$ and $\mathcal{D}_F$ are parallel. They were introduced in the first half of the last century by Schouten and Van Kampen [7] and Vrănceanu [8] for studying the geometry of a non-holonomic space. In the modern terminology a non-holonomic space is a manifold endowed with a non-integrable distribution. The coordinate-free expressions for the Schouten–Van Kampen connection ∇ and the Vrănceanu connection ∇^* were given by Ianuş [3] as follows:

$$\nabla_X Y = V \widetilde{\nabla}_X VY + H \widetilde{\nabla}_X HY, \tag{2.8}$$

and

$$\nabla^*_X Y = V \widetilde{\nabla}_{VX} VY + H \widetilde{\nabla}_{HX} HY + V[HX, VY] + H[VX, HY], \tag{2.9}$$

respectively, for any $X, Y \in \Gamma(TM)$, where H and V are the projection morphisms of $\Gamma(TM)$ on $\Gamma(\mathcal{D})$ and $\Gamma(\mathcal{D}_F)$ respectively. By using (2.4), (2.8), (2.9) and (2.7c) we deduce that

$$\begin{aligned} &\text{(a) } \nabla_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_{\alpha}{}^{\gamma}{}_{\beta} \frac{\delta}{\delta x^\gamma}, \qquad \text{(b) } \nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial y^i} = K_{i}{}^{j}{}_{\gamma} \frac{\partial}{\partial y^j}, \\ &\text{(c) } \nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^\alpha} = L_{\alpha}{}^{\gamma}{}_{i} \frac{\delta}{\delta x^\gamma}, \qquad \text{(d) } \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = P_{i}{}^{k}{}_{j} \frac{\partial}{\partial y^k}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\text{(a) } \nabla^*_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_{\alpha}{}^{\gamma}{}_{\beta} \frac{\delta}{\delta x^\gamma}, \qquad \text{(b) } \nabla^*_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial y^i} = D_{i}{}^{j}{}_{\alpha} \frac{\partial}{\partial y^j}, \\ &\text{(c) } \nabla^*_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^\alpha} = 0, \qquad \text{(d) } \nabla^*_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = P_{i}{}^{k}{}_{j} \frac{\partial}{\partial y^k}. \end{aligned} \tag{2.11}$$

By using ∇ and ∇^* we can define two types of covariant derivatives for an adapted tensor field on M. Namely, the horizontal and the vertical covariant derivatives of $T = \left(T^{i\alpha}_{j\beta}\right)$ induced by Schouten–Van Kampen connection are defined by

$$T^{i\alpha}_{j\beta|\gamma} = \frac{\delta T^{i\alpha}_{j\beta}}{\delta x^\gamma} + T^{h\alpha}_{j\beta} K_{h}{}^{i}{}_{\gamma} + T^{i\varepsilon}_{j\beta} F_{\varepsilon}{}^{\alpha}{}_{\gamma} - T^{i\alpha}_{h\beta} K_{j}{}^{h}{}_{\gamma} - T^{i\alpha}_{j\varepsilon} F_{\beta}{}^{\varepsilon}{}_{\gamma}, \tag{2.12}$$

and

$$T^{i\alpha}_{j\beta\|k} = \frac{\partial T^{i\alpha}_{j\beta}}{\partial y^k} + T^{h\alpha}_{j\beta} P_{h}{}^{i}{}_{k} + T^{i\varepsilon}_{j\beta} L_{\varepsilon}{}^{\alpha}{}_{k} - T^{i\alpha}_{h\beta} P_{j}{}^{h}{}_{k} - T^{i\alpha}_{j\varepsilon} L_{\beta}{}^{\varepsilon}{}_{k}, \tag{2.13}$$

respectively.

Similarly, the horizontal and vertical covariant derivatives induced by the Vrănceanu connection are given by

$$T^{i\alpha}_{j\beta\,|^*\gamma} = \frac{\delta T^{i\alpha}_{j\beta}}{\delta x^\gamma} + T^{h\alpha}_{j\beta} D_{h}{}^{i}{}_{\gamma} + T^{i\varepsilon}_{j\beta} F_{\varepsilon}{}^{\alpha}{}_{\gamma} - T^{i\alpha}_{h\beta} D_{j}{}^{h}{}_{\gamma} - T^{i\alpha}_{j\varepsilon} F_{\beta}{}^{\varepsilon}{}_{\gamma}, \tag{2.14}$$

and

$$T^{i\alpha}_{j\beta\,\|^*k} = \frac{\partial T^{i\alpha}_{j\beta}}{\partial y^k} + T^{h\alpha}_{j\beta} P_{h}{}^{i}{}_{k} - T^{i\alpha}_{h\beta} P_{j}{}^{h}{}_{k}, \tag{2.15}$$

respectively. It is noteworthy that all these covariant derivatives define adapted tensor fields on M.

Now, we recall that the Levi-Civita connection $\widetilde{\nabla}$ on (M, g) is given by (cf. O'Neill [4], p.61)

$$\begin{aligned} 2g(\widetilde{\nabla}_X Y, Z) = {} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \end{aligned} \tag{2.16}$$

for any $X, Y, Z \in \Gamma(TM)$.

Theorem 2.1. *The Levi-Civita connection on an oblique warped product* $(M = E \times F_{(f,L)}, g)$ *is completely determined by the following local coefficients:*

$$
\begin{aligned}
&\text{(a)}\ F_{\alpha}{}^{\gamma}{}_{\beta} = \frac{1}{2} g^{\gamma\mu} \left(\frac{\delta g_{\mu\alpha}}{\delta x^{\beta}} + \frac{\delta g_{\mu\beta}}{\delta x^{\alpha}} - \frac{\delta g_{\alpha\beta}}{\delta x^{\mu}} \right), \\
&\text{(b)}\ G_{\alpha}{}^{i}{}_{\beta} = -\frac{1}{2} \left(\widetilde{g}^{ik} g_{\alpha\beta\,\|^*k} + I_{\alpha}{}^{i}{}_{\beta} \right), \\
&\text{(c)}\ K_{i}{}^{j}{}_{\alpha} = \frac{1}{2} \widetilde{g}^{jk} \left(\widetilde{g}_{ik\,|^*\alpha} + 2\, D_{i}{}^{h}{}_{\alpha}\, \widetilde{g}_{hk} \right), \\
&\text{(d)}\ N_{i}{}^{\gamma}{}_{j} = -\frac{1}{2} g^{\gamma\alpha}\, \widetilde{g}_{ij\,|^*\alpha}, \\
&\text{(e)}\ P_{i}{}^{h}{}_{j} = \frac{1}{2} k^{h\ell} \left(\frac{\partial k_{\ell i}}{\partial y^{j}} + \frac{\partial k_{\ell j}}{\partial y^{i}} - \frac{\partial k_{ij}}{\partial y^{\ell}} \right).
\end{aligned}
\tag{2.17}
$$

Proof. The main tools for the proof are (2.16), (2.7a), (2.7c), (2.1) and the above covariant derivatives with respect to Vrănceanu connection. First, we take $X = \frac{\delta}{\delta x^{\beta}}$, $Y = \frac{\delta}{\delta x^{\alpha}}$, $Z = \frac{\delta}{\delta x^{\mu}}$ in (2.16) and by using (2.7a) and the first equality in (2.1a) we obtain (2.17a). If we take the same X, Y but $Z = \frac{\partial}{\partial y^{j}}$, then (2.16) yields (2.17b) via (2.1a), (2.1b), (2.7a) and (2.15). The other three formulas are obtained in a similar way. Finally, by using (2.7g), (2.7h) and (2.7i) we deduce that the coefficients in (2.17) completely determine $\widetilde{\nabla}$. □

Lemma 2.3. (i) *The vertical covariant derivatives of* g_{ij} *and* $g_{\alpha\beta}$ *with respect to the Schouten–Van Kampen connection and Vrănceanu connection are given by*

$$\text{(a)}\ \widetilde{g}_{ij\|k} = 0, \qquad \text{(b)}\ g_{\alpha\beta\|k} = 0, \tag{2.18}$$

$$\text{(a)}\ \widetilde{g}_{ij\,\|^*k} = 0, \qquad \text{(b)}\ g_{\alpha\beta\,\|^*k} = \frac{1}{2} \widetilde{g}_{ki} \left(G_{\alpha}{}^{i}{}_{\beta} + G_{\beta}{}^{i}{}_{\alpha} \right), \tag{2.19}$$

respectively.

(ii) *The horizontal covariant derivatives of* g_{ij} *and* $g_{\alpha\beta}$ *with respect to Schouten–Van Kampen connection and Vrănceanu connection are given by*

$$\text{(a)}\ \widetilde{g}_{ij|\gamma} = 0, \qquad \text{(b)}\ g_{\alpha\beta|\gamma} = 0, \tag{2.20}$$

$$\text{(a)}\ \widetilde{g}_{ij\,|^*\gamma} = -2\, N_{i}{}^{\varepsilon}{}_{j}\, g_{\varepsilon\gamma}, \qquad \text{(b)}\ g_{\alpha\beta\,|^*\gamma} = 0, \tag{2.21}$$

respectively.

Proof. The main tool in the proof is (2.6). First, (2.18a) and (2.19a) follow from (2.6) on taking $X = \frac{\partial}{\partial y^{k}}$, $Y = \frac{\partial}{\partial y^{i}}$, $Z = \frac{\partial}{\partial y^{j}}$ and using (2.1c), (2.4d), (2.13) and (2.15). In a similar way we obtain (2.18b), (2.20) and (2.21b). Next, (2.19b) follows from (2.17b) on using (2.7f) for the integrability tensor of $\mathcal{D}$. Finally, (2.21a) is a consequence of (2.17d). □

From (2.18) and (2.20) we see that the Schouten–Van Kampen connection is a metric connection. In contrast, (2.19b) and (2.21a) show that, in general, the Vrănceanu connection is not a metric connection. We say that ∇^* is a *vertical* (resp. *horizontal*) *metric connection* if we have

$$g_{\alpha\beta\,\|^*k} = 0 \quad (\text{resp. } \widetilde{g}_{ij\,|^*\gamma} = 0). \tag{2.22}$$

To relate this to the geometry of M we recall some special classes of foliations. If all leaves of a foliation are totally geodesic (resp. totally umbilical) we say that the foliation is totally geodesic (resp. totally umbilical). Now consider a foliation $\mathcal{F}$ on a semi-Riemannian manifold (M, g) and denote by $\mathcal{H}$ the transversal distribution to $\mathcal{F}$, that is, $\mathcal{H}$ is the complementary orthogonal distribution to the tangent distribution to $\mathcal{F}$ in TM. If each geodesic in (M, g) which is tangent to $\mathcal{H}$ at one point remains tangent for its entire length, we say that g is bundle-like for $\mathcal{F}$ (cf. Reinhart [5]).

Now, denote by $\mathcal{F}_F$ the foliation with tangent distribution $\mathcal{D}_F$ and call it the vertical foliation on the oblique warped product M. Then, from (2.4d) we deduce that $\mathcal{F}_F$ is totally geodesic if and only if

$$N_i{}^\gamma{}_j = 0, \quad \forall \gamma \in \{1, \ldots, p\}, i, j \in \{1, \ldots, n\}. \tag{2.23}$$

Finally, according to Reinhart [6], p.156, we deduce that g is bundle-like for $\mathcal{F}_F$ if and only if

$$\frac{\partial g_{\alpha\beta}}{\partial y^k} = 0, \quad \forall \alpha, \beta \in \{1, \ldots, p\}, k \in \{1, \ldots, n\}. \tag{2.24}$$

Now, we can prove the following.

Theorem 2.2. *Let $(M = E \times F_{(f,L)}, g)$ be an oblique warped product and $\mathcal{F}_F$ be the vertical foliation on M. Then we have the assertions:*

(i) *g is bundle-like for $\mathcal{F}_F$ if and only if the Vrănceanu connection is a vertical metric connection.*

(ii) *$\mathcal{F}_F$ is totally geodesic if and only if the Vrănceanu connection is a horizontal metric connection.*

Proof. By the definition of the vertical covariant derivative with respect to Vrănceanu connection (see (2.15)) we deduce that

$$g_{\alpha\beta\,\|^*k} = \frac{\partial g_{\alpha\beta}}{\partial y^k}\,. \tag{2.25}$$

Then the assertion (i) follows by using (2.25), (2.24) and (2.22). The assertion (ii) is a consequence of (2.21)–(2.23). □

Corollary 2.1. *The vertical foliation on an oblique warped product is totally geodesic with bundle-like metric if and only if the Vrănceanu connection is a metric connection.*

We denote also by k the lift of the semi-Riemannian metric k from F to $\mathcal{D}_F$. Then we say that k is *horizontal Vrănceanu parallel* if we have

$$k_{ij\,|^*\gamma} = 0, \quad \forall \gamma \in \{1, \ldots, p\}, i, j \in \{1, \ldots, n\}. \tag{2.26}$$

Proposition 2.1. *If k is horizontal Vrănceanu parallel then the vertical foliation $\mathcal{F}_F$ is totally umbilical.*

Proof. By using (2.1c) we obtain

$$\widetilde{g}_{ij\,|^*\gamma} = k_{ij\,|^*\gamma}(f \circ \pi)^2 + 2k_{ij} f \frac{\partial f}{\partial x^\gamma}\,.$$

Then, taking into account (2.26) and (2.21a), we deduce that

$$N_i{}^\varepsilon{}_j = -k_{ij} f g^{\varepsilon\gamma} \frac{\partial f}{\partial x^\gamma},$$

that is, $\mathcal{F}_F$ is totally umbilical. □

Next, we denote by T and T^* the torsion tensor fields of ∇ and ∇^* respectively. Then we have

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad \forall X, Y \in \Gamma(TM), \tag{2.27}$$

and a similar formula for T^*. Now we put

$$\begin{aligned} &(a) \quad T\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = T_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + T_\alpha{}^i{}_\beta \frac{\partial}{\partial y^i}, \\ &(b) \quad T\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i}\right) = T_i{}^\gamma{}_\alpha \frac{\delta}{\delta x^\beta} + T_i{}^j{}_\alpha \frac{\partial}{\partial y^j}, \qquad (c) \quad T\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = T_i{}^\gamma{}_j \frac{\delta}{\delta x^\gamma} + T_i{}^k{}_j \frac{\partial}{\partial y^k}\,. \end{aligned} \tag{2.28}$$

The local components of T^* are defined in the same way as above, but they will have a star.

Lemma 2.4. (i) *The local components of the torsion tensor field of the Schouten–Van Kampen connection are given by*

$$\begin{aligned} &\text{(a) } T_{\alpha}{}^{\gamma}{}_{\beta}=0, \quad \text{(b) } T_{\alpha}{}^{i}{}_{\beta}=I_{\alpha}{}^{i}{}_{\beta}, \quad \text{(c) } T_{i}{}^{\gamma}{}_{\alpha}=-L_{\alpha}{}^{\gamma}{}_{i}, \\ &\text{(d) } T_{i}{}^{j}{}_{\alpha}=M_{\alpha}{}^{j}{}_{i}, \quad \text{(e) } T_{i}{}^{\gamma}{}_{j}=0, \quad \text{(f) } T_{i}{}^{k}{}_{j}=0. \end{aligned} \tag{2.29}$$

(ii) *The local components of the torsion tensor field of the Vrănceanu connection are given by*

$$\begin{aligned} &\text{(a) } T^{*}{}_{\alpha}{}^{\gamma}{}_{\beta}=0, \quad \text{(b) } T^{*}{}_{\alpha}{}^{i}{}_{\beta}=I_{\alpha}{}^{i}{}_{\beta}, \quad \text{(c) } T^{*}{}_{i}{}^{\gamma}{}_{\alpha}=0, \\ &\text{(d) } T^{*}{}_{i}{}^{j}{}_{\alpha}=0, \quad \text{(e) } T^{*}{}_{i}{}^{\gamma}{}_{j}=0, \quad \text{(f) } T^{*}{}_{i}{}^{k}{}_{j}=0. \end{aligned} \tag{2.30}$$

Proof. First, by using (2.27), (2.10a), (2.7a) and (2.7e) we obtain

$$T\left(\frac{\delta}{\delta x^{\beta}}, \frac{\delta}{\delta x^{\alpha}}\right) = I_{\alpha}{}^{i}{}_{\beta}\, \frac{\partial}{\partial y^{i}},$$

which implies both (2.29a) and (2.29b). Next, we use (2.28), (2.10b), (2.10c), (2.7c), (2.7h), and deduce that

$$T\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial y^{i}}\right) = M_{\alpha}{}^{j}{}_{i}\, \frac{\partial}{\partial y^{j}} - L_{\alpha}{}^{\gamma}{}_{i}\, \frac{\delta}{\delta x^{\gamma}},$$

which yields (2.29c) and (2.29d) via (2.28b). Finally, (2.29e) and (2.29f) follow from (2.26) by using (2.10d) and (2.28c). This completes the proof of assertion (i). The assertion (ii) is proved in a similar way. □

By combining Lemma 2.2 with the assertion (ii) of Lemma 2.4 we obtain the following.

Corollary 2.2. *The (f, k)-associated distribution on an oblique warped product is integrable if and only if the Vrănceanu connection is torsion-free.*

From the assertion (i) of Lemma 2.4 we see that the Schouten–Van Kampen connection is far from being torsion-free. To see the structure of the manifold M in this case we give the following definition. We say that the oblique warped product (M, g) is a locally semi-Riemannian product if the (f, k)-associated distribution $\mathcal{D}$ is integrable and the foliations tangent to $\mathcal{D}$ and $\mathcal{D}_F$ are totally geodesic. Now we prove the following.

Theorem 2.3. *Let (M, g) be an oblique warped product. Then the following assertions are equivalent:*

(i) *M is a locally semi-Riemannian product.*
(ii) *The Schouten–Van Kampen connection is torsion-free.*
(iii) *The Schouten–Van Kampen and Vrănceanu connections coincide.*

Proof. (i) $\Longrightarrow$ (ii). Since $\mathcal{D}$ is integrable, by Lemma 2.2 and (2.29b) we deduce that $T_{\alpha}{}^{i}{}_{\beta}=0$. Taking into account that the leaves of both distributions $\mathcal{D}$ and $\mathcal{D}_F$ are totally geodesic immersed in (M, g), from (2.4a) and (2.4d) we obtain $G_{\alpha}{}^{i}{}_{\beta}=0$ and $N_{i}{}^{\gamma}{}_{j}=0$, for all $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta, \gamma \in \{1, \ldots, p\}$. Then (2.7g), (2.7i) and (2.7j) imply $L_{\alpha}{}^{\gamma}{}_{i}=0$ and $M_{\alpha}{}^{j}{}_{i}=0$. Thus by (2.29c) and (2.29d) we infer that $T_{i}{}^{\gamma}{}_{\alpha}=0$ and $T_{i}{}^{j}{}_{\alpha}=0$. Hence the Schouten–Van Kampen connection is torsion-free.

(ii) $\Longrightarrow$ (iii). Since ∇ is torsion-free, from (2.29c) and (2.29d) we obtain $L_{\alpha}{}^{\gamma}{}_{i}=0$ and $M_{\alpha}{}^{j}{}_{i}=0$. Then (2.10b) and (2.10c) become

$$\nabla_{\frac{\delta}{\delta x^{\alpha}}} \frac{\partial}{\partial y^{i}} = D_{i}{}^{j}{}_{\alpha}\, \frac{\partial}{\partial y^{j}} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial y^{i}}} \frac{\delta}{\delta x^{\alpha}} = 0, \tag{2.31}$$

respectively, via (2.7h). Finally, we compare (2.10) and (2.11) taking into account (2.31) and obtain $\nabla = \nabla^{*}$.

(iii) $\Longrightarrow$ (i). If $\nabla = \nabla^{*}$, then from (2.10) and (2.11) by using (2.7h) we obtain $L_{\alpha}{}^{\gamma}{}_{i}=0$ and $M_{\alpha}{}^{j}{}_{i}=0$. Then (2.7g), (2.7i) and (2.7j) imply $I_{\alpha}{}^{i}{}_{\beta}=0$, $G_{\alpha}{}^{i}{}_{\beta}=0$ and $N_{i}{}^{\gamma}{}_{j}=0$. Thus $\mathcal{D}$ is integrable and the leaves of $\mathcal{D}$ and $\mathcal{D}_F$ are totally geodesic immersed in (M, g). This completes the proof of the theorem. □

Remark 2.1. Any of the above assertions is true if and only if $\nabla = \nabla^{*} = \widetilde{\nabla}$. Indeed, if (ii) is true then ∇ is torsion-free and it is a metrical connection. So by the uniqueness of the Levi-Civita connection we have $\nabla = \widetilde{\nabla}$. Conversely, if $\nabla = \widetilde{\nabla}$, then ∇ should be torsion-free, so (ii) is satisfied. □

3. Curvature of an oblique warped product

Let $\widetilde{\nabla}$ and ∇ be the Levi-Civita and Schouten–Van Kampen connections on the oblique warped product $(M = E \times F_{(f,L)}, g)$. Then by using (2.8) and taking into account the decomposition (2.3) we put

$$\widetilde{\nabla}_X VY = \nabla_X VY + B(X, VY), \tag{3.1}$$

and

$$\widetilde{\nabla}_X HY = B'(X, HY) + \nabla_X HY, \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$, where B and B' are given by

$$\text{(a) } B(X, VY) = H\widetilde{\nabla}_X VY \quad \text{and} \quad \text{(b) } B'(X, HY) = V\widetilde{\nabla}_X HY. \tag{3.3}$$

By using (3.1)–(3.3) and (2.6) we obtain

$$g(B(X, VY), HZ) + g(B'(X, HZ), VY) = 0, \tag{3.4}$$

for any $X, Y, Z \in \Gamma(TM)$.

The Schouten–Van Kampen connection enables us to define some covariant derivatives for B and B' as follows:

$$(\nabla_X B)(Y, VZ) = \nabla_X(B(Y, VZ)) - B(\nabla_X Y, VZ) - B(Y, \nabla_X VZ), \tag{3.5}$$

and

$$(\nabla_X B')(Y, HZ) = \nabla_X(B'(Y, HZ)) - B'(\nabla_X Y, HZ) - B'(Y, \nabla_X HZ), \tag{3.6}$$

for any $X, Y, Z \in \Gamma(TM)$. Now, we denote by R and $\widetilde{R}$ the curvature tensor fields of ∇ and $\widetilde{\nabla}$ and state the following.

Theorem 3.1. *The curvature tensor fields of the Levi-Civita and Schouten–Van Kampen connections satisfy the following equations:*

$$g(\widetilde{R}(X, Y)VZ, VU) = g(R(X, Y)VZ, VU) + g(B(X, VZ), B(Y, VU)) - g(B(Y, VZ), B(X, VU)), \tag{3.7}$$

$$g(\widetilde{R}(X, Y)VZ, HU) = g((\nabla_X B)(Y, VZ) - (\nabla_Y B)(X, VZ), HU) + g(B(T(X, Y), VZ), HU), \tag{3.8}$$

$$\begin{aligned} g(\widetilde{R}(X, Y)HZ, HU) = {} & g(R(X, Y)HZ, HU) + g(B'(X, HZ), B'(Y, HU)) \\ & - g(B'(Y, HZ), B'(X, HU)), \end{aligned} \tag{3.9}$$

$$g(\widetilde{R}(X, Y)HZ, VU) = g((\nabla_X B')(Y, HZ) - (\nabla_Y B')(X, HZ), VU) + g(B'(T(X, Y), HZ), VU), \tag{3.10}$$

for any $X, Y, Z \in \Gamma(TM)$, where T is the torsion field of ∇.

Proof. By using (3.1) and (3.2) we obtain

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y VZ = \nabla_X \nabla_Y VZ + B(X, \nabla_Y VZ) + B'(X, B(Y, VZ)) + \nabla_X(B(Y, VZ)). \tag{3.11}$$

On the other hand, (3.1) and (2.27) imply

$$\widetilde{\nabla}_{[X,Y]} VZ = \nabla_{[X,Y]} VZ + B(\nabla_X Y, VZ) - B(\nabla_Y X, VZ) - B(T(X, Y), VZ). \tag{3.12}$$

Thus by using (3.11), (3.12) and (3.5) we deduce that

$$\begin{aligned} \widetilde{R}(X, Y)VZ &= [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]VZ - \widetilde{\nabla}_{[X,Y]} VZ \\ &= \{R(X, Y)VZ + B'(X, B(Y, VZ)) - B'(Y, B(X, VZ))\} \\ &\quad + \{(\nabla_X B)(Y, VZ) - (\nabla_Y B)(X, VZ) + B(T(X, Y), VZ)\}. \end{aligned} \tag{3.13}$$

Now, we take the $\mathcal{D}$- and $\mathcal{D}_F$-components in (3.13) and obtain (3.8) and

$$g(\widetilde{R}(X, Y)VZ, VU) = g(R(X, Y)VZ, VU) + g(B'(X, B(Y, VZ)) - B'(Y, B(X, VZ)), VU). \tag{3.14}$$

Finally, by using (3.4) in (3.14) we obtain (3.7). By similar calculations we obtain (3.9) and (3.10). □

It is important to note that (3.8) and (3.10) are equivalent. This follows by direct calculations using (3.4) and properties of $\widetilde{R}$. As the study of the geometry of the foliation $\mathcal{F}_F$ reduces in many respects to the geometry of its leaves, we concentrate our attention on the (f, k)-associate distribution $\mathcal{D}$. Let $z \in M$ and W be a 2-dimensional subspace of $\mathcal{D}_z$ which we call a $\mathcal{D}$-plane. Take a basis $\{u, v\}$ of W and define

$$\Delta(u, v) = g(u, u)g(v, v) - g(u, v)^2.$$

Then W is non-degenerate if and only if $\Delta(u, v) \neq 0$. Next, we define the number

$$K(u, v) = \frac{g(R(u, v)v, u)}{\Delta(u, v)}, \tag{3.15}$$

provided W is non-degenerate. Taking into account that the Schouten–Van Kampen connection is a metric connection we deduce that $K(u, v)$ given by (3.15) is independent of the basis $\{u, v\}$. Then we denote it by $K(W)$ and call it the *Schouten–Van Kampen sectional curvature* of $\mathcal{D}$ at z with respect to the non-degenerate $\mathcal{D}$-plane W.

Example 3.1. Let $I \times S^3_{(f,L)}$ be the g.o.w. product obtained for $q = 1$ in Example 1.2. Then by the method we developed in Example 1.4 we obtain oblique warped product structures on $M = I^2 \times S^3$. Here we describe explicitly one of these structures and calculate the Schouten–Van Kampen sectional curvature of the distribution $\mathcal{D}$. To this end we take (x^α, y^i) as local coordinates on M, where (x^α), $\alpha \in \{1, 2\}$, and (y^i), $i \in \{1, 2, 3\}$, are coordinates on I^2 and S^3, respectively. Then we consider on M the metric g which, with respect to the natural frame field $\left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^i}\right\}$ is given by the matrix

$$\begin{bmatrix} f(x^2) & 0 & 0 \\ 0 & -1 & \eta_j (f \circ \pi)^2 \\ 0 & \eta_i (f \circ \pi)^2 & k_{ij} (f \circ \pi)^2 \end{bmatrix}.$$

Thus we have

$$L_{1i} = 0 \quad \text{and} \quad L_{2i} = \eta_i (f \circ \pi)^2,$$

which via (1.18) yield

$$D^i_1 = 0 \quad \text{and} \quad D^i_2 = \xi^i.$$

Then by (1.7) we deduce that the orthogonal complementary distribution $\mathcal{D}$ to $\mathcal{D}_{S^3}$ in TM is locally spanned by

$$\left\{\frac{\delta}{\delta x^1} = \frac{\partial}{\partial x^1}, \frac{\delta}{\delta x^2} = \frac{\partial}{\partial x^2} - \xi^i \frac{\partial}{\partial y^i}\right\}.$$

Next, by using (2.1a) and taking into account that $\xi = (\xi^i)$ is a unit vector field on S^3, we obtain

$$[g_{\alpha\beta}] = \begin{bmatrix} f(x^2) & 0 \\ 0 & -(1 + f^2(x^2)) \end{bmatrix}.$$

Then by direct calculations using (2.10a) and (2.17a) we deduce that

$$\nabla_{\frac{\delta}{\delta x^1}} \frac{\delta}{\delta x^2} = \nabla_{\frac{\delta}{\delta x^2}} \frac{\delta}{\delta x^1} = \frac{f'}{2f} \frac{\delta}{\delta x^1},$$
$$\nabla_{\frac{\delta}{\delta x^1}} \frac{\delta}{\delta x^1} = \frac{f'}{2(1 + f^2)} \frac{\delta}{\delta x^2},$$
$$\nabla_{\frac{\delta}{\delta x^2}} \frac{\delta}{\delta x^2} = \frac{ff'}{1 + f^2} \frac{\delta}{\delta x^2}.$$

Taking into account that

$$\left[\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}\right] = 0,$$

and by using the above formulas for the Schouten–Van Kampen connection we obtain

$$R\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}\right)\frac{\delta}{\delta x^1} = \frac{(f')^2(1+3f^2) - 2ff''(1+f^2)}{4f(1+f^2)^2}\,\frac{\delta}{\delta x^2}\,.$$

Finally, we have

$$\Delta\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}\right) = -f(1+f^2),$$

and by using (3.15) we deduce that

$$K\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}\right) = \frac{(f')^2(1+3f^2) - 2ff''(1+f^2)}{4f^2(1+f^2)^2}\,.$$

For any f with $f'' < 0$ we obtain a large class of oblique warped products with positive Schouten–Van Kampen sectional curvatures for the (f, k)-associated distributions. □

To define a sectional curvature of $\mathcal{D}$ with respect to Vrănceanu connection ∇^* we need a study of its curvature tensor field R^*. This is because ∇^*, in general, is not a metric connection (see Corollary 2.1). First, we observe that $G_{\alpha}{}^{i}{}_{\beta}$ from (2.4a) are the local components for

$$B' : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}_F) : B'(HX, HY) = V\widetilde{\nabla}_{HX}HY. \tag{3.16}$$

Then by (2.19) and the assertion (i) of Theorem 2.2 we deduce the following.

Theorem 3.2. *The semi-Riemannian metric g on the oblique warped product M is bundle-like for the vertical foliation $\mathcal{F}_F$ if and only if we have*

$$B'(HX, HY) + B'(HY, HX) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{3.17}$$

Also, we prove the following.

Lemma 3.1. *Let (M, g) be an oblique warped product where g is bundle-like for $\mathcal{F}_F$. Then the curvature tensor fields R and R^* satisfy the identity*

$$R(HX, HY)HZ = R^*(HX, HY)HZ - 2B(HZ, B'(HX, HY)), \tag{3.18}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. By using (2.8), (2.9) and (3.3a) we deduce that

$$\nabla_X HZ = \nabla^*_X HZ + B(HZ, VX), \quad \forall X, Z \in \Gamma(TM). \tag{3.19}$$

Then by direct calculations using (3.19) we obtain

$$R(HX, HY)HZ = R^*(HX, HY)HZ - B(HZ, V[HX, HY]). \tag{3.20}$$

Next, by using (2.5), (3.3b) and (3.17), we infer that

$$V[HX, HY] = V\widetilde{\nabla}_{HX}HY - V\widetilde{\nabla}_{HY}HX = 2B'(HX, HY). \tag{3.21}$$

Thus (3.18) follows from (3.20) by using (3.21). □

Lemma 3.2. *Let (M, g) as in Lemma 3.1. Then the curvature tensor field of Vrănceanu connection satisfies the identity*

$$g(R^*(HX, HY)HZ, HU) + g(R^*(HX, HY)HU, HZ) = 0, \tag{3.22}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. By using (3.18) and (3.4) we obtain

$$g(R(HX, HY)HZ, HU) = g(R^*(HX, HY)HZ, HU) + 2g(B'(HZ, HU), B'(HX, HY)). \tag{3.23}$$

Taking into account that R satisfies such an identity (3.22) (since ∇ is a metric connection), and by using (3.17) in (3.23) we obtain (3.22). □

By using properties of R^* (including (3.22)) we define the *Vrănceanu sectional curvature* $K^*(W)$ of $\mathcal{D}$ at z with respect to the non-degenerate $\mathcal{D}$-plane W by (3.15), but with R^* instead of R. Similarly, we have $\widetilde{K}(W)$ given by (3.15), but with $\widetilde{R}$ instead of R. These sectional curvatures are related as follows.

Theorem 3.3. *Let (M, g) be an oblique warped product, where g is bundle-like for $\mathcal{F}_F$. Then the Schouten–Van Kampen, Vrănceanu and Levi-Civita sectional curvatures of the (f, k)-associated distribution $\mathcal{D}$ are related by*

$$3K(W) = 2\widetilde{K}(W) + K^*(W), \tag{3.24}$$

for any non-degenerate $\mathcal{D}$-plane W.

Proof. Let $\{HX, HY\}$ be a basis of W. Then by using (3.9) and (3.17) we obtain

$$g(\widetilde{R}(HX, HY)HY, HX) = g(R(HX, HY)HY, HX) - g(B'(HX, HY), B'(HX, HY)),$$

which implies

$$\widetilde{K}(W) = K(W) - \frac{g(B'(HX, HY), B'(HX, HY))}{\Delta(HX, HY)}. \tag{3.25}$$

On the other hand, by using (3.18), (3.4) and (3.17) we deduce that

$$g(R(HX, HY)HY, HX) = g(R^*(HX, HY)HY, HX) - 2g(B'(HX, HY), B'(HX, HY)),$$

which yields

$$K(W) = K^*(W) - 2\,\frac{g(B'(HX, HY), B'(HX, HY))}{\Delta(HX, HY)}. \tag{3.26}$$

Thus (3.24) follows from (3.25) and (3.26). □

Corollary 3.1. *Let (M, g) be an oblique warped product, where g is a Riemannian metric that is bundle-like for $\mathcal{F}_F$. Then we have*

$$\widetilde{K}(W) \le K(W) \le K^*(W). \tag{3.27}$$

Proof. In this case we have

$$\Delta(HX, HY) > 0 \quad \text{and} \quad g(B'(HX, HY), B'(HX, HY)) \ge 0.$$

Then (3.27) follows from (3.25) and (3.26). □

In particular, when the ambient manifold (M, g) is of non-negative sectional curvatures, then both the Schouten–Van Kampen and Vrănceanu sectional curvatures are non-negative too.

4. Oblique Robertson–Walker spacetime

As is well known, the standard models of the universe are warped products. They are the simplest models of neighborhoods of stars and black holes. According to the theory we developed in this paper we may think of generalizations of these models. Here we present such a generalization for the Robertson–Walker spacetime.

Let $M = I \times S$, where I is an open interval in $\mathbb{R}$ and S is a connected 3-dimensional manifold. Choose the coordinates (t, y^1, y^2, y^3), where t is the parameter on I and (y^1, y^2, y^3) are the local coordinates on S. Then $U = \frac{\partial}{\partial t}$ gives the velocity of each galaxy $\gamma_y(t) = (t, y)$, where $y \in S$, and $\left\{\frac{\partial}{\partial y^i}\right\}$, $i \in \{1, 2, 3\}$, is the natural frame field on S. The standard Robertson–Walker model was obtained by imposing some assumptions on the galactic flow. Following O'Neill [4], p. 342 we present them here:

(a) The semi-Riemannian metric g on M satisfies

$$g(U, U) = -1.$$

(b) For any $t \in I$, each slice $S(t) = \{t\} \times S$ has U as normal vector. In other words, the leaves $I \times \{y\}$ and fibres $\{t\} \times S$ must be orthogonal with respect to g.
(c) The isotropy condition "all spatial directions the same" stated as follows: Each (t, y) has a neighborhood $\mathcal{V}$ in M, such that given unit tangent vectors v, v' to $S(t)$ at (t, y), there is a galaxy-preserving isometry $\varphi = id \times \varphi_{|S}$ such that $d\varphi(v) = v'$.

It follows that S must be a Riemannian manifold of constant curvature $c = -1, 0$ or 1. Then the warped product

$$M(c, f) = I \times_f S,$$

where f is a positive smooth function on I, is called a Robertson–Walker spacetime (cf. O'Neill [4], p. 343).

Now, we follow the above ideas, but we omit the condition (b). Therefore the leaves and fibres of $M = I \times S$ are no longer orthogonal with respect to g. In other words, taking into account the notation from this paper we have the following geometric objects: a negative definite metric h on I, a Riemannian metric k on S, a positive smooth function f on I and a non-zero $\mathcal{F}(M)$-bilinear mapping $L : \Gamma(\mathcal{D}_I) \times \Gamma(\mathcal{D}_S) \longrightarrow \mathcal{F}(M)$, such that g given by (see (1.2))

$$g(X, Y) = h(P_I X, P_I Y) + (f \circ \pi)^2 k(P_S X, P_S Y) + L(P_I X, P_S Y) + L(P_I Y, P_S X), \tag{4.1}$$

is a Lorentz metric on M. By Theorem 1.2, given f, h, k, there exists $L \neq 0$ such that g is a Lorentz metric on M. As the condition (c) still holds, the fibres $S(t)$ are of constant curvature $c = -1, 0$ or 1. Thus the oblique warped product

$$M(c, f, L) = I \times S_{(f,L)},$$

is called an *oblique Robertson–Walker spacetime*. It is important to note that in this case, the (f, k)-associated distribution $\mathcal{D}$, being of rank 1, is integrable. Denote by $\mathcal{F}$ the foliation tangent to $\mathcal{D}$. Then we state the following.

Proposition 4.1. *Let $(M(c, f, L), g)$ be an oblique Robertson–Walker spacetime such that g is bundle-like for the vertical foliation $\mathcal{F}_S$ (the foliation by fibres $S(t), t \in I$). Then the foliation $\mathcal{F}$ is totally geodesic.*

Proof. In this case (2.19b) becomes

$$g_{11\,\|^* k} = \widetilde{g}_{ki} G^i_{11}. \tag{4.2}$$

Then by assertion (i) of Theorem 2.2 and (4.2) we deduce that $G^i_{11} = 0, i \in \{1, 2, 3\}$. Hence the assertion follows from (2.4a). □

Remark 4.1. Here, and in the following formulas we use the general theory from previous sections but all Greek indices are equal to 1, and $i, j, k, \ldots \in \{1, 2, 3\}$. □

Next, by using (1.7), (1.18) and (1.10) we deduce that the (f, k)-associated distribution $\mathcal{D}$ is given by

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - D^i_1 \frac{\partial}{\partial y^i}, \tag{4.3}$$

where the D^i_1 are given by

$$D^i_1 = L_{1j} k^{ji} \frac{1}{(f \circ \pi)^2}, \qquad L_{1j} = g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y^j}\right).$$

Thus the local components of the Lorentz metric g with respect to the semi-holonomic frame field $\left\{\frac{\delta}{\delta t}, \frac{\partial}{\partial y^i}\right\}$ become (see (2.1))

$$\text{(a) } g_{11}(t, y) = -1 - D^i_1 D^j_1 k_{ij}(y) f^2(t), \qquad \text{(b) } g'_{1i} = 0, \qquad \text{(c) } \widetilde{g}_{ij}(t, y) = k_{ij}(y) f^2(t). \tag{4.4}$$

According to the study we developed in Section 2, we may state the following (see Theorem 2.2 and Proposition 2.1).

Theorem 4.1. *The vertical foliation on an oblique Robertson–Walker spacetime is totally geodesic if and only if the Vrănceanu connection is a horizontal metric connection.*

Proposition 4.2. *If the lift of the Riemannian metric k from S to $I \times S$ is horizontal Vrănceanu parallel then the vertical foliation on an oblique Robertson–Walker spacetime is totally umbilical.*

Next, for some standard choices of S we can construct oblique warped product structures on $I \times S$ by the general method described in Example 1.3. Thus when we take the unit sphere S^3 as S, we can consider on $I \times S^3$ the oblique warped product structure presented in Example 1.2. The same construction applies to $\mathbb{R}^3$ and to the hyperbolic space H^3 as well. For all these cases we have

$$D_1^i = \xi^i, \tag{4.5}$$

where (ξ^i) are the local components of a unit vector field ξ on S. Thus (4.4) becomes

$$\text{(a) } g_{11}(t) = -1 - f^2(t), \qquad \text{(b) } g'_{1i} = 0, \qquad \text{(c) } \widetilde{g}_{ij}(t, y) = f^2(t)k_{ij}(y). \tag{4.6}$$

Hence we have

$$\text{(a) } g^{11}(t) = -\frac{1}{1 + f^2(t)}, \qquad \text{(b) } \widetilde{g}^{ij}(t, y) = \frac{1}{f^2(t)}\, k^{ij}(y). \tag{4.7}$$

Then by using the vertical covariant derivative induced by the Vrănceanu connection (see (2.15)) we obtain

$$g_{11\|^* k} = 0, \quad \forall k \in \{1, 2, 3\}. \tag{4.8}$$

Proposition 4.3. *The (f, k)-associated distribution $\mathcal{D}$ determined by (4.5) on the oblique Robertson–Walker spacetime $I \times S_{(f,L)}$ is totally geodesic.*

Proof. Taking into account that $\mathcal{D}$ is integrable, that is, $I_1{}^i{}_1 = 0, i = \{1, 2, 3\}$, and by using (4.8) in (2.17b) we obtain $G_1{}^i{}_1 = 0, i \in \{1, 2, 3\}$. Thus the assertion follows from (2.4a). □

Now, by using (2.7d) and (4.5) we deduce that

$$D_i{}^j{}_1 = \frac{\partial \xi^j}{\partial y^i} = \xi^j{}_{|i} - \xi^k P_k{}^j{}_i, \tag{4.9}$$

where the covariant derivative is taken with respect to the Levi-Civita connection on S. Then by direct calculations using (2.14), (4.6), (4.9) and (2.19a) we obtain

$$\begin{aligned} \widetilde{g}_{ij\,|^*1} &= \frac{\delta \widetilde{g}_{ij}}{\delta t} - \widetilde{g}_{hj}\, D_i{}^h{}_1 - \widetilde{g}_{ih}\, D_j{}^h{}_1 = \frac{\partial \widetilde{g}_{ij}}{\partial t} - \xi^k \frac{\partial \widetilde{g}_{ij}}{\partial y^k} - \widetilde{g}_{hj}(\xi^h{}_{|i} - \xi^k P_k{}^h{}_i) - \widetilde{g}_{ih}(\xi^h{}_{|j} - \xi^k P_k{}^h{}_j) \\ &= 2f(t)f'(t)k_{ij}(y) - f^2(t)\{k_{hi}\xi^h{}_{|j} + k_{hj}\xi^h{}_{|i}\}. \end{aligned} \tag{4.10}$$

Proposition 4.4. *The foliation determined by fibres of the oblique Robertson–Walker spacetime $I \times S^3_{(f,L)}$ given by (4.5) is totally umbilical.*

Proof. First, we note that ξ is a Killing vector field on S^3 (cf. Yano and Kon [9], p. 275), that is, we have

$$k_{hi}\xi^h{}_{|j} + k_{hj}\xi^h{}_{|i} = 0.$$

Thus (4.10) becomes

$$\widetilde{g}_{ij\,|^*1} = 2f(t)f'(t)k_{ij}(y).$$

Then from (2.17d) we deduce that

$$N_i{}^1{}_j = -\frac{f(t)f'(t)}{1 + f^2(t)}\, k_{ij}(y).$$

Finally, our assertion follows from (2.4d), since ${N_i}^1{}_j$ are the local components of the second fundamental forms of the fibres $S^3(t)$. □

The properties stated in Propositions 4.3 and 4.4 say that the geometry of an oblique Robertson–Walker spacetime is still close to what is known for the usual Robertson–Walker spacetime. More results on both the geometry and physics of an oblique Robertson–Walker spacetime will appear in a forthcoming paper.

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